**HANDOUT with TRANSCRIPTS: Differentiating Instruction on Speed for Middle School Students**

**Five Practices for Differentiating Mathematics Instruction** (Hackenberg, Creager, & Eker, 2021):
1. Using research-based knowledge of students' mathematical thinking
2. Providing purposeful choices and different pathways
3. Inquiring responsively during group work
4. Attending to small group functioning
5. Conducting whole class discussions across different thinkers

**Research-based knowledge on units coordination**

| UC stage | Solving Vacation Problem | Result | Multiplication & Division | Measurement |
|---|---|---|---|---|
| 1 | *Track 7s with units of 5 <br> *Count on by 1s past known skip-counting patterns | *35 is a composite unit and 35 singleton units, no multiplicative relationship between 1 and 35 | *Not inversely related | *Connected number where unit is not multiplicatively related to other numbers |
| 2 | *Track 7s with units of 5 <br> *Take out 7s and break apart to add them strategically | *35 is a composite unit with anticipated multiplicative rel. between 1 and 35 | *Not inversely related | *Connected number where unit is multiplicatively related to other numbers |
| 3 | *Explicitly embed units of 7 in units of 5 <br> *Can reason by breaking apart each 7 into 5 and 2 | *35 is a composite unit with anticipated multiplicative rel. between 7 and 35, or 5 and 35 | *Inversely related | *Connected number where fractions of unit are mult. related to other numbers |

**Speed Investigation:**
- middle part of a 26-day unit on proportional reasoning
- 18 7th grade students, fall semester
- Used "races" geogebra app: https://www.geogebra.org/m/hc3ecnnt

**Making red car go slower…**
- Black car travels 15 miles in 6 min. Red car travels 15 miles in ___ min. Find a number of minutes that makes the red car go slower.
- Black car travels 15 miles in 6 min. Red car travels ___ miles in 6 min. Find a number of minutes that makes the red car go slower.

**Same Speed Task:**
Black car goes ___ miles in ___ minutes.
Make the red car go the same speed but a different number of minutes and miles. Draw a picture to explain/justify.

**Differentiating Plan**

| UC stage | Orangeyness Tasks | Planned Numbers for Same Speed Task | Rationale |
|---|---|---|---|
| 1 | *Not iterating two quantities as a composed unit | 18 mi in 3 min | Whole number unit ratio (6 mi in 1 min) |
| 2 | *Iterating two quantities as a composed unit | 15 mi in 6 min | Mixed number unit ratio with ½ (2.5 mi in 1 min) |
| 3 | *Iterating two quantities as a composed unit<br>*Partitioning two quantities to get a unit ratio | 15 mi in 9 min | Unit ratio hard to work with as a decimal (5/3 mi in 1 min) |

**Project website:** https://education.indiana.edu/research/initiatives/IDReAM/index.html

**EMILY**

**Transcript for Day 12, Emily & Group, Solving:** Black car goes 18 mi in 3 min

EMILY: Okay. So, for the blue car...

LENA: I can't see. I want to see. Go like this. [They angle the ipad so everyone can see.]

EMILY: 18, 3. [She types this into the app.] What numbers do you think we can use to make it go the same speed?

LENA: The same speed?

EMILY: Yeah.

LENA: 9 and 2.

EMILY: 9 and 2? [She types this into the app.]

LENA: I don't know. I'm just guessing. Or 9 and 1. Put in 1 minute. That'd be funny.

EMILY: Maybe 9 and 6… because times 2.

[They watch the cars race.]

LENA [laughs]: The blue car definitely won.

EMILY: Oh, I know! Oh, 18 and 6. [She runs the app and does not seem satisfied.] No. 18 and 2! [She runs the app and does not seem satisfied.] Oh. Never mind. [Turns to Kendra] What number do you think, Kendra?

KENDRA: I don't know.

EMILY: We're just making a prediction, like it doesn't really matter.

[Lena is writing something on her paper.]

LENA: Ooh, try 36 and 6.

[Emily puts this into the app and runs it.]

EMILY: Oh, oh, oh! [She seems very excited.]


**Transcript for Day12, Emily & Group, Justifying/Drawing Picture:** Black car goes 18 mi in 3 min

AH:     Now, take one of those and see if you can explain why it is, like why does 36 miles in 6 minutes, why is that the same speed as 18 miles in 3 minutes? Can you draw a picture to show? [pause] Try to draw a picture that would convince somebody, because somebody might look at that and be like, how do you know that? How would you convince them? [10-second pause]

EMILY: I don't know.

AH:     Can you draw a picture that shows something about the journeys? You've got this 18 mile in 3 minute journey, right, 18 miles in 3 minutes—that's a journey. You could represent that somehow by, I don't know. How would you represent that, 18 miles in three minutes? [pause]

EMILY: I don't know. It's not like I can just draw a picture in a way. It's kind of like… I don't really know how to show it. I know how to tell, but I don't know how to show it.

AH:     Okay, how would you tell? What would you say?

EMILY: Well, 18, we did 18 times 2 and 3 times 2, and they're the same because [pause] you can reduce it. You can divide it. You can divide 36 by 2 and get 18, and you can divide 6 by 2 and get 3. So, it's almost like they're the same number in a way.

AH:     Uh-huh, there's something that's the same about it, about those pairs?

EMILY: Yeah.

AH:     That's great, but now I want us to go a little deeper than that because you're thinking about the numbers, but I want you to think about the quantities, the miles and the minutes, because that's where you're going to have the most strength in developing something that's going to help you think about speed in general. So, that's why I'm asking for a picture. So, you said something about multiplying by 2. Is there any way to show that in a picture, like to show how 18 miles in 3 minutes and 36 miles in 6 minutes is like, 36 miles in 6 minutes is twice 18 miles in 3 minutes. Can you just draw something to kind of show that idea?

EMILY [overlapping with AH]: What you could do, is you could do. Well, I don't really… can I write
down the numbers, like can I write down the numbers?

AH:     Sure, yeah.

EMILY [writing]: So, 18 miles in 3 minutes.

AH:     Think about drawing a journey of the car, to help you think about how to get a picture going.

EMILY: Then it's 18 times 2 and then you get 36 and 6. [Emily shows multiplying each number by 2
with notation.]

AH:     Right, but I still want to know how do you know that has to be the same speed as this? I know
you guys are convinced because you've checked it on the app, but we've got to go a little bit
deeper than that. We've got to try to justify why. Why does doubling produce the same speed? If
you just doubled the miles, would it be the same speed?

EMILY: No.

AH:     If you just doubled time, would it be the same speed?

EMILY: No.

AH:     No, so there's something important about doubling both of them, and that's what you're trying to
explain. So, how could you draw a picture of 18 miles in 3 minutes? How could you just draw--
you don't have to draw a car--you don't have to draw anything other than something to show 18
miles in 3 minutes. What could show it? [7-second pause] What if you want to show, say hey
somebody, the car went 18 miles and it happened in 3 minutes, what would you draw to show the
car's journey?

EMILY [draws a car]: That's a bad car. But you have a car.

AH: Uh-huh.

EMILY: And it's going [draws a line next to the car, ending at a house with "3 min" written over the
line].

AH:     Okay. Okay, great. I noticed, Emily, you have 3 minutes there. It's kind of like you're saying this
is this distance that got covered in 3 minutes. What's the distance again?

[Emily writes "18 mil" under the line.]

AH:     All right, super. So, right here you have this segment, or line, that would show that distance.
Now, we also have the other car going 36 miles in 6 minutes. Do you think you could show that
one?

[Emily draws another line segment below the first. She writes 6 min above the line and 36 miles below
the line.]

**Emily's first picture:**
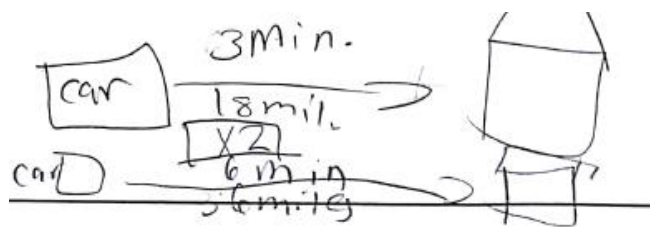


AH:     Now, would these be the same length?

EMILY: No.

AH:     What's the relationship between the length of this one and the length of that one?

EMILY: That one's [top one] shorter than that one [bottom one].

AH:     It's shorter, okay. Can you tell me anything more? How much shorter—I mean right now yours is
shorter [on the paper]. Do you think it shows well that this is 18 and that's 36?

EMILY: Yeah.

AH:     How come?

EMILY: Because it's labeling them.

AH:     **Do you have the idea of doubling in there though, that you were talking about?**

[Emily writes "x 2" between the two drawings.]

AH:    **Okay, well can you show it with the lengths? Emily, can you show the doubling of the lengths? Do you get what I'm asking about?**

EMILY: It's two times larger.

AH:    Yeah. **Can you draw it that way?** Maybe draw it again over here. We can use this, just so you can really see. Do you guys, Kendra and Lena, do you understand what we're talking about? She's drawing a length to show one journey, and then she's saying the other one is two times larger, so she's going to try it to try to show the two times larger. [Someone sneezes at another table.] Bless you. What do you think, Lena?

LENA: I don't know.

[Emily draws a second picture.]

**Emily's second picture:**
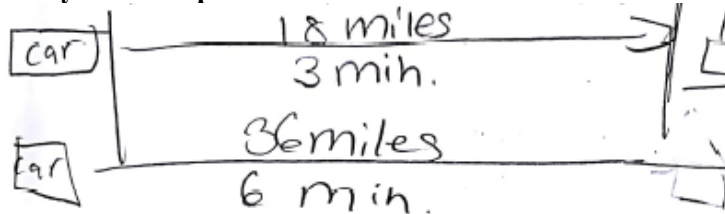


AH [to Emily]: **That's a really nice picture. To me, though, when I look at it, it looks like this journey and that journey are the same size [the two line segments are the same size]. Are they the same size?**

EMILY: No.

AH:    Something's the same about them, the speed. **Is there any way to show how the one journey is, as you said, twice as long, twice as big?** [Emily extends the segment showing 36 miles in 6 min, but she does not make it twice as big as the segment showing the 18 miles in 3 min.] Oh, okay, that's, now does it show it exactly? I wonder if there's a way to show it pretty exactly.

[Emily uses her fingers to measure off the 18 mile-3 min segment and moves it over, drawing another segment. This second segment is shorter.]

**Emily's third picture:**
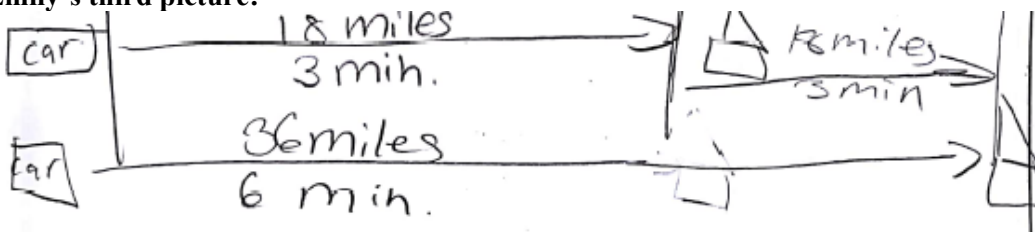


AH:    Pretty good? Do you think it's pretty good? You want these two? Are these two just supposed to be the same size [the two segments on the top of the drawing]?

EMILY: Yeah, but not drawn to scale, but yeah.

AH:    Yeah, they're supposed to be the same size? **Do you think you could draw a picture where you show them the same size?** I think that would really help because it shows what this journey [36 miles in 6 min] is made out of. Do you see what I'm saying? This journey, it looks like you're saying, Emily, is made out of going 18 miles in 3 minutes, and then doing it again.

EMILY: Yeah, it is!

**LISA & SARA**
**Transcript for Lisa & Sara, Breakthrough:** Black car goes 15 mi in 6 min

LISA:  It's impossible, even when we do that, 15.1 and 6.1, because it's not 15 or 6.
AH:       Okay, so you really think it's impossible?
LISA & SARA: Yeah.
SARA: Yeah, unless you do...
AH:       **So, two cars can't go the same exact speed but go different distances and times?**
SARA: They probably could, but I can't figure it out.
LISA:   When you say we can't use 15 or 6, it's kind of hard.
AH:       Right. All right, well that's good to know that it's hard.
SARA: Unless you double it, and it's going the same speed.
AH:       What do you mean, Sara?
LISA: 30 and 12.
AH:       **Try that.** What does that give you?
SARA: 30 and 12?
LISA:   Yeah, it's doubling.
AH:       Double the distance and double the time. [SARA: I don't know.] Do you think you'll go the same
          speed or no?
SARA: I mean they're not going the same exact speed but they're going the same speed, just…
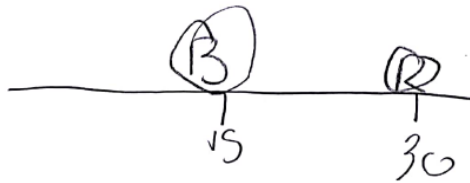AH:       Do you think it'll be the same speed, not the same exact?
SARA: I figured the system out!
LISA:   Wow, it worked. Okay.

**Transcript for Lisa & Sara, Explaining Doubling:** Black car goes 15 mi in 6 min

[Sara has stated that doubling makes the same speed. AH asks why. She also asks about drawing a
picture. Lisa has started a picture.]

**Lisa's initial picture:**



AH:       Lisa, **one thing I see in your picture is the distance from 0 to 15, right? Is this 0 right here?**
LISA:   Yeah.
AH:       **Okay, and then onto 30. So, it looks to me like you're saying if you go 30 miles, it's like you
          go 15 and then another…**
LISA:   15.
AH:       **15, right. So, then I wonder if that can be a beginning to show why you know that when you
          double, it goes the same speed.** You and Sara should definitely talk. Then you guys have another
          challenge.
LISA:   Uuuuh [groaning].
AH:       Once you get the picture done, there's another challenge.
LISA:   We already did get the picture done.

AH:    Well, but can you explain to me how that shows that 30 miles in 12 minutes is the same as 15 miles in 6?

SARA: I just don't, I can't process a picture like, this is the best I can do [pointing to her picture, which is similar to Lisa's above], and I have to multiply both of them by two in order to get double, to make it go the same speed. If you double it, they're going the same exact speed, just they're going a farther distance. One's going a farther distance than the other.

AH:    That's good, okay. So, you've got farther distance. Does time have anything to do with it?

SARA: I mean yeah. I mean, well…

AH:    Why do you have to double the time?

SARA: Well...

AH [To Sara]: **See if you can involve time in your picture.**

LISA:  So, this took 6 minutes and this took another 6 minutes.

AH:    Okay.

LISA:  We add that.

AH:    I see. So, Lisa, in your picture, where is the whole journey of 30 miles in 12 minutes? That's 30 miles gone in 12 minutes, so that'd be like this whole thing, right?

LISA:  Yeah.

AH:    Then it's made from one 15-mile 6-minute journey and another 15-mile and 6 -minute journey? [Lisa starts to say this too.]

LISA:  Yes.


**Transcript for Lisa & Sara, Making Smaller Distance-Time Pairs:** Black car goes 15 mi in 6 min


LISA:  Do this one, 7.5...

RB [research team member]: So, Lisa, tell Sara why you think that'll work and where those numbers came from.

LISA:  Because I divided it by 2 .

RB:    Don't tell me. Tell Sara.

SARA: Why is it divided by 2?

LISA:  So, that it would, because she said divide it, make it smaller, so 15 divided by 2, and make that one go 15 in 6 minutes, and do the Ferrari in 3 minutes. [Sara is putting this into the app.]

SARA: Yeah, they're going the same time.

LISA:  Yeah, I was right.

SARA: So, it doesn't matter if it's double the speed or minus the speed [Lisa: divided]. It just has to be equal or whatever.

RB:    Okay, **so Lisa and Sara, when you doubled, that was like taking a trip twice, right?**

LISA:  Yeah.

RB:    **So, what you just did, what is that like doing?**

LISA:  Dividing.

SARA: Taking a trip minus one. [laughs]

RB:    What do you mean minus one? Does that mean 1 mile less?

LISA:  No. You divide by 2 .

RB:    So, what does that mean, you divide by 2?

LISA:  That means you'll go dividing, you'll go… uh…

SARA: I just feel like it's going...

LISA:  You'll go 2 miles less or something like that.

RB:    So, is 7.5 two miles less than 15?

SARA: Yes.

LISA:  No!

SARA: No. [To Lisa, loudly] I don't know what you did!

LISA:  Wait. [Counts on fingers] 7, 8, 9, 10, 11, 12, 13, 14, 8 1/2 miles less.

RB:	Okay. I don't think it was 8 1/2 miles less.
LISA:	Then 7 1/2 miles less.
RB:	Okay, so 7 1/2 miles less?
LISA:	Yeah.
RB:	Put that in your picture. Can you draw a picture of what you just did?
LISA:	On this side, because we already did that picture?
RB:	Maybe see if you can put it in relation to this picture. So, if this is your 15 and 3 and this is your 30 and 6, is it 15 and 6? 15 and 6 and 30 and 12, where would 7.5 and 3 be?
[Short bit of time where students drew.]

**Lisa's picture now:**
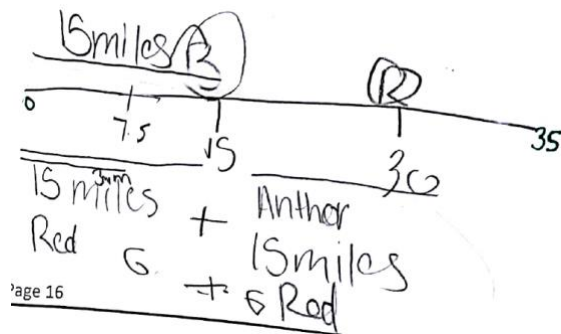


RB:	**So, instead of doubling, you…**
LISA:	Did dividing.
RB:	**You divided it but you divided it by 2, which is something kind of special, right?**
LISA:	Because it's not even.
SARA:	If you divide it by 3, it's not even.
LISA:	You times that one by 2 and…
RB:	**What would happen if we divide it by three?**
LISA:	I don't know, I don't know.
SARA:	15 divided by 3…
LISA:	Oh, it's 5.
SARA:	So, 5, and what's 6 divided by 3?
LISA:	One, no 2.
SARA:	2?
LISA:	I think. [Looks at RB.]
SARA:	Right. [Tries it on app.] WHAT?!! Oh yeah, yeah, yeah. [The race finishes and both girls look up in surprise.]
LISA:	It worked!!
SARA:	Why does it work?
RB:	It worked?
LISA and SARA: Yes.
LISA:	Maybe because it's a factor of 15.
RB:	You should record that in your chart.

**JOANNA**
**Transcript for Day12, Joanna & Group, Solving:** Black car goes 15 mi in 9 min

MARK: Wouldn't 16 and 10 work?
JOANNA: I don't think it would.
MARK: Why not?
JOANNA: Well, the 16 and 10, see 15 and 9 reduced would be 5 to 3, but 16…
MARK: So, it has to go [inaudible].
JOANNA: It would be 5, like a ratio would be 5 to 3, and then for the red one if you did 16 and 9, 16 and 10, that would reduce to 8 to 5, and that's not 5 to 3. No, it's not.
BRIANNE: I want to test something.
JOANNA: So, does it just have to be, sorry [to Mark]. You go.
MARK: I don't really know how to explain it but, do you get what I'm saying by 16 and 10? Because it'd be—
JOANNA: Yeah, I do but it still wouldn't [work]. Sorry.
MARK: Of course. Why would I be right? I'm never right.
JOANNA: I'm sorry. No, no. I just...
MARK: Joanna is always right and here to always ruin my amazing brilliant ideas.
JOANNA [to Brianne and Jenni]: Do you agree with him? Because…
MARK: At least she doesn't want to hurt my feelings.
JOANNA: Because they wouldn't be the same ratio to each other.
MARK: That's fine.
JOANNA: No, I'm sorry.
BRIANNE [to Mark]: You want to see if it works, 16 and 10?  [They test it in the app.]
MARK: Mm-hmm [yes].
BRIANNE: The red car is slightly behind the blue car.
MARK: Oh, man.
JOANNA: Oh, man.
BRIANNE: Okay, I think that 30 and 18 will work, yeah.
JENNI: That's what I was just thinking.
JOANNA: Do you know what I think would work? Any two numbers that will reduce to 5 and 3.
BRIANNE: 15?
JOANNA: So, 10 and 6.
MARK: Wait. What's your idea?
JOANNA: Any numbers where the miles would reduce to five and the minutes would reduce to three, because it'd be the same ratio to each other.

**Transcript for Day12, Joanna Explaining Picture:** Black car goes 15 mi in 9 min

JOANNA: I kind of did a number line and I was showing how much the blue car went, which is 15 miles in 9 minutes. And I divided it up into 3 [parts], so it's 0 and then it goes 5 and 10, 15, and this goes 0, 3, to 6, to 9. And I was just showing that it goes the same distance because this goes this much and this much in the same speed. So they go the same speed; this one just stops earlier.
RB:    So, it looks like you are saying this whole thing is that one trip of 15 miles in 9 minutes?
JOANNA: Yes.
RB:    **So what would this little piece be?**
JOANNA: That'd be 5 miles in 3 minutes.
RB:    **So how does that relate to the whole trip?**
JOANNA: 5 miles in 3 minutes would be one-third of the trip.
RB:    A third of the trip?

JOANNA: Yes.
RB:      **So if you do that third three times you end up with the whole trip?**
JOANNA: Yes.

**Joanna's picture:**